

Approximating the Schur multiplier of certain infinitely presented groups via nilpotent quotients

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Abstract

We describe an algorithm for computing successive quotients of the Schur multiplier $M(G)$ for a group G given by an invariant finite L -presentation. As application, we investigate the Schur multipliers of various self-similar groups such as the Grigorchuk super-group, the generalized Fabrykowski-Gupta groups, the Basilica group and the Brunner-Sidki-Vieira group.

Keywords: Schur multiplier; recursive presentations; Grigorchuk group; self-similar groups;

1 Introduction

The Schur multiplier $M(G)$ of a group G can be defined as the second homology group $H_2(G, \mathbb{Z})$. It was introduced by Schur and is, for instance, relevant in the theory of central group extensions. In combinatorial group theory, the Schur multiplier found its applications due to the Hopf formula: if F is a free group and R is a normal subgroup of F so that $G \cong F/R$, then the Schur multiplier of G is isomorphic to the factor group $(R \cap F')/[R, F]$. For further details on the Schur multiplier we refer to [24, Chapter 11].

The Hopf formula yields that every finitely presentable group has a finitely generated Schur multiplier. This is used in [15] for proving that the Grigorchuk group is not finitely presentable: its Schur multiplier is infinitely generated 2-elementary abelian. This answers the questions in [6] and [25]. There are various examples of self-similar groups other than the Grigorchuk group for which it is not known whether their Schur multiplier is finitely generated or whether the groups are finitely presented.

The first aim of this paper is to introduce an algorithm for investigating the Schur multiplier of self-similar groups with a view towards its finite generation. Let G be a group with a presentation $G \cong F/R$. Then $G/\gamma_c G \cong F/R\gamma_c F$, where $\gamma_c G$ is the c -th term of the lower central series of G . We identify $M(G)$ with $(R \cap F')/[R, F]$ and $M(G/\gamma_c G)$ with $(R\gamma_c F \cap F')/[R\gamma_c F, F]$ and define

$$\varphi_c: M(G) \rightarrow M(G/\gamma_c G), \quad g[R, F] \mapsto g[R\gamma_c F, F].$$

Then φ_c is a homomorphism of abelian groups. We describe an effective method to determine the *Dwyer quotients* $M_c(G) = M(G)/\ker \varphi_c$, for $c \in \mathbb{N}$,

provided that G is given by an invariant finite L -presentation, see [1, 2] or Section 2 below. Every finitely presented group and many self-similar groups can be described by a finite invariant L -presentation. An implementation of our algorithm is available in the NQL-package [19] of the computer algebra system GAP; see [12].

We have applied our algorithm to various examples of self-similar groups: the Grigorchuk super-group $\tilde{\mathfrak{G}}$, see [3], the Basilica group Δ , see [18, 17], the Brunner-Sidki-Vieira group BSV, see [9], and some generalized Fabrykowski-Gupta groups Γ_d , see [11, 16]. As a result, we observed that the sequence $(M_1(G), \dots, M_c(G), M_{c+1}(G), \dots)$ exhibits a periodicity in c in all these cases. Based on this, we propose the following conjecture.

Conjecture I

- $M_c(\tilde{\mathfrak{G}})$ is 2-elementary abelian of rank $2\lfloor \log_2(c) \rfloor + 2\lfloor \log_2 \frac{c}{3} \rfloor + 5$, for $c \geq 4$.
- $M_c(\Delta)$ has the form $\mathbb{Z}^2 \times \mathcal{A}_c$, where \mathcal{A}_c is an abelian 2-group of rank $\lfloor \log_2 \frac{c}{3} \rfloor$ and exponent $2^{2\lfloor \frac{c-6}{2} \rfloor + 2}$, for $c \geq 6$.
- $M_c(\text{BSV})$ has the form $\mathbb{Z}^2 \times \mathcal{B}_c$, where \mathcal{B}_c is an abelian 2-group of rank $\lfloor \log_2 \frac{c}{3} \rfloor + \lfloor \log_2 \frac{c}{9} \rfloor + 3$ and exponent $2^{2\lfloor \frac{c-4}{2} \rfloor + 1}$, for $c \geq 4$.
- For a prime power d , the group $M_c(\Gamma_d)$ has exponent d for c large enough; its rank is an increasing function in c which exhibits a periodic pattern.

In particular, all of these groups have an infinitely generated Schur multiplier and are therefore not finitely presentable.

Further details on the periodicities and the computational evidence for them are given in Section 6.

2 Preliminaries

In the following, we recall the basic notion of invariant and finite L -presentations and the basic theory of the Schur multiplier of a group. Let F be a finitely generated free group over the alphabet \mathcal{X} . Further suppose that $\mathcal{Q}, \mathcal{R} \subset F$ are finite subsets of the free group F and $\Phi \subset \text{End}(F)$ is a finite set of endomorphisms of F . Then the quadruple $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ is a *finite L -presentation*. It defines the *finitely L -presented group*

$$G = \left\langle \mathcal{X} \mid \mathcal{Q} \cup \bigcup_{\varphi \in \Phi^*} \mathcal{R}^\varphi \right\rangle,$$

where Φ^* denotes the free monoid generated by Φ ; that is, the closure of $\Phi \cup \{\text{id}\}$ under composition. A finite L -presentation $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ is *invariant* if every endomorphism $\varphi \in \Phi$ induces an endomorphism of G ; that is, if the normal closure of $\mathcal{Q} \cup \bigcup_{\varphi \in \Phi^*} \mathcal{R}^\varphi$ in F is φ -invariant. For example, every finite L -presentation of the form $\langle \mathcal{X} \mid \emptyset \mid \Phi \mid \mathcal{R} \rangle$ is invariant. Clearly, invariant finite L -presentations generalize finite presentations since every finitely presented

group $\langle \mathcal{X} \mid \mathcal{R} \rangle$ is finitely L -presented by $\langle \mathcal{X} \mid \emptyset \mid \{\text{id}\} \mid \mathcal{R} \rangle$. Further examples of invariantly L -presented groups are several self-similar groups including the Grigorchuk group [14], the Basilica group [18, 17], and the Brunner-Sidki-Vieira group [9].

In the remainder of this section, we recall the basic theory of the Schur multiplier of a group G . Recall that, in general, the Schur multiplier of a finitely presented group is not computable; see [13]. But, for instance, if G is finite, then $M(G)$ can be deduced from a finite presentation of G with the Hopf formula and the Reidemeister-Schreier algorithm. A more effective algorithm for finite permutation groups is described in [20]. Recently, Eick and Nickel [10] described an algorithm for computing the Schur multiplier of a polycyclic group given by a polycyclic presentation.

Let F be a free group and R be a normal subgroup of F so that $G \cong F/R$. Then the Hopf formula gives

$$M(G) \cong (R \cap F')/[R, F]. \quad (1)$$

Suppose that N is a normal subgroup of G and let S be a normal subgroup of F so that SR/R corresponds to N . Then Blackburn and Evens [7] determined the exact sequence

$$1 \rightarrow (R \cap [S, F])/([R, F] \cap [S, F]) \rightarrow M(G) \rightarrow M(G/N) \rightarrow (N \cap G')/[N, G] \rightarrow 1.$$

Applying this sequence to the lower central series term $N = \gamma_c G$ yields the exact sequence

$$1 \rightarrow (R \cap \gamma_{c+1} F)/([R, F] \cap \gamma_{c+1} F) \rightarrow M(G) \xrightarrow{\varphi_c} M(G/\gamma_c G) \rightarrow \gamma_c G/\gamma_{c+1} G \rightarrow 1.$$

This gives a filtration $M(G) \geq \ker \varphi_1 \geq \ker \varphi_2 \geq \dots$, called the *Dwyer-filtration*, of the Schur multiplier of G . Note that, if G has a maximal nilpotent quotient of class c , then

$$\bigcap_{c \in \mathbb{N}_0} \ker \varphi_c \cong (R \cap \gamma_{c+1} F)[R, F]/[R, F].$$

However, even if the group G is residually nilpotent, the group $F/[R, F]$ is not necessarily residually nilpotent; see [22] and [8]. Thus the *Dwyer-kernel* $\bigcap_{c \in \mathbb{N}} \ker \varphi_c$ is possibly non-trivial.

We note that the Schur multiplier $M(G/\gamma_c G)$ can be computed with the algorithm in [10] while the isomorphism type of $\gamma_c G/\gamma_{c+1} G$ can be computed with the nilpotent quotient algorithm in [2]. Therefore, the sequence $M(G) \rightarrow M(G/\gamma_c G) \rightarrow \gamma_c G/\gamma_{c+1} G \rightarrow 1$ allows to determine the size of $M_c(G)$ provided that $M(G/\gamma_c G)$ is finite. However, the algorithm described here determines the structure of $M_c(G)$ even if the Schur multiplier $M(G/\gamma_c G)$ is infinite.

3 Adjusting an invariant L -presentation

In order to prove the following theorem, we explicitly describe an algorithm for modifying an invariant L -presentation. The resulting L -presentation enables us to read off a generating set for the Schur multiplier in Section 4. Our algorithm generalizes the explicit computations in [15].

Theorem 1 *Let $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ be an invariant finite L -presentation which defines the group $G = F/R$. Then G admits an invariant finite L -presentation $\langle \mathcal{X} \mid \mathcal{Q}' \cup \mathcal{B} \mid \Phi \mid \mathcal{R}' \rangle$ with $\mathcal{Q}', \mathcal{R}' \subset F'$ and $\mathcal{B} \subset F$ satisfying $|\mathcal{B}| = |\mathcal{X}| - \mathfrak{h}(G/G')$, where $\mathfrak{h}(G/G')$ denotes the torsion-free rank of G/G' .*

Proof. Since $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ is an invariant L -presentation, every endomorphism $\varphi \in \Phi$ induces an endomorphism of the group G . Thus we have $R^\varphi \subseteq R$, for every $\varphi \in \Phi^*$. In particular, every image of a relator in $\mathcal{Q} \cup \mathcal{R}$ is a consequence; that is, $\mathcal{Q}^\varphi \subset R$ and $\mathcal{R}^\varphi \subset R$, for every $\varphi \in \Phi^*$.

Write $n = \text{rk}(F)$. Then the abelianization $\pi: F \rightarrow \mathbb{Z}^n$ maps every $x \in F$ to its corresponding exponent vector $a_x \in \mathbb{Z}^n$. Clearly, $\ker \pi = F'$ and, since F' is fully-invariant, every $\varphi \in \Phi$ induces an endomorphism of the free abelian group \mathbb{Z}^n . Therefore, the exponent vector of x^φ is the image $a_x M_\varphi$ for some matrix $M_\varphi \in \mathbb{Z}^{n \times n}$. Now, the normal subgroup RF' maps onto

$$U = \langle a_q, a_r M_\varphi \mid q \in \mathcal{Q}, r \in \mathcal{R}, \varphi \in \Phi^* \rangle \leq \mathbb{Z}^n. \quad (2)$$

As every subgroup of \mathbb{Z}^n is generated by at most n elements, the subgroup U is finitely generated. In the following, we may use the spinning algorithm from [2] and Hermite normal form computations to compute a basis for the subgroup U while modifying the L -presentation simultaneously.

Let B be a basis of $\langle a_q \mid q \in \mathcal{Q} \rangle$. Then every element $u \in B$ is a \mathbb{Z} -linear combination of elements in $\{a_q \mid q \in \mathcal{Q}\}$. Hence, for every $u \in B$, there exists a word r_u in the relators in \mathcal{Q} such that $a_{r_u} = u$. Define $\mathcal{B} = \{r_u \mid u \in B\}$. Then, for every $q \in \mathcal{Q}$, it holds that $a_q \in \langle B \rangle$ as B is a basis and hence, there exists a word w_q in the r_u 's so that $a_{w_q} = a_q$. Define $\mathcal{Q}' = \{qw_q^{-1} \mid q \in \mathcal{Q}\}$. Then the exponent vector of each element in \mathcal{Q}' vanishes and hence $\mathcal{Q}' \subset F'$. Moreover, the invariant and finite L -presentation

$$\langle \mathcal{X} \mid \mathcal{Q}' \cup \mathcal{B} \mid \Phi \mid \mathcal{R} \rangle$$

still defines the group G as we only applied Tietze transformations to the given L -presentation.

It remains to force the elements of \mathcal{R} into the derived subgroup F' . For this purpose, we will use the spinning algorithm from [2] as follows: Initialize $\mathcal{R}' = \emptyset$. As long as \mathcal{R} is non-empty, we take an element $r \in \mathcal{R}$ and remove it from \mathcal{R} . Then either $a_r \in \langle B \rangle$ or $a_r \notin \langle B \rangle$ holds. If $a_r \in \langle B \rangle$, then there exists a word w_r in the r_u 's such that $a_{w_r} = a_r$ and hence, $rw_r^{-1} \in F'$. In this case we just add rw_r^{-1} to \mathcal{R}' . Note that, for every $\varphi \in \Phi^*$, the word $(w_r^{-1})^\varphi$ is a consequence and hence, we can replace the relator r^φ in the L -presentation by $(rw_r^{-1})^\varphi$. The invariant and finite L -presentation

$$\langle \mathcal{X} \mid \mathcal{Q}' \cup \mathcal{B} \mid \Phi \mid \mathcal{R}' \cup \mathcal{R} \rangle$$

still defines the group G .

If, on the other hand, $a_r \notin \langle B \rangle$ holds, we enlarge the current basis B and modify the set \mathcal{B} . Let B' be a basis for $\langle B \cup \{a_r\} \rangle$. Then every $v \in B'$ is a \mathbb{Z} -linear combination of the elements in $B \cup \{a_r\}$ and hence, there exists a word \tilde{r}_v in $\mathcal{B} \cup \{r\}$ such that $a_{\tilde{r}_v} = v$. Define $\mathcal{B} = \{\tilde{r}_v \mid v \in B'\}$. Then, by

construction, either $|\mathcal{B}| = |B| + 1$ or $|\mathcal{B}| = |B|$ holds. In the latter case, there is an element $u \in B$ so that $u \in \langle (B \setminus \{u\}) \cup \{a_r\} \rangle$ holds. Thus, there exists a word w_u in the elements of \mathcal{B} such that $a_{w_u} = u$ and hence, $r_u w_u^{-1} \in F'$. In this case, we add $r_u w_u^{-1}$ to \mathcal{Q}' and add the images $\{r_u^\varphi \mid \varphi \in \Phi\}$ to \mathcal{R} . This yields an invariant and finite L -presentation $\langle \mathcal{X} \mid \mathcal{Q}' \cup \mathcal{B} \mid \Phi \mid \mathcal{R}' \cup \mathcal{R} \rangle$, with $\mathcal{Q}', \mathcal{R}' \subset F'$, which still defines the group G .

As ascending chains of subgroups in \mathbb{Z}^n terminate, eventually every exponent vector of an element in \mathcal{R} is contained in the subgroup $\langle B \rangle$ and hence, the algorithm described above eventually terminates. Clearly, the basis B is then a basis for the subgroup U in (2). As shown in [2], the abelian quotient G/G' is isomorphic to the factor \mathbb{Z}^n/U . Its torsion-free rank is $n - |B|$ as claimed above. \square

In the following example, we recall the explicit computations in [15] for the Grigorchuk group \mathfrak{G} .

Example 2 Consider the Grigorchuk group \mathfrak{G} with its invariant L -presentation

$$\mathfrak{G} \cong \langle \{a, b, c, d\} \mid \{a^2, b^2, c^2, d^2, bcd\} \mid \{\sigma\} \mid \{(ad)^4, (adacac)^4\} \rangle$$

where σ is the free group endomorphism induced by the mapping

$$\sigma: \begin{cases} a \mapsto c^a \\ b \mapsto d \\ c \mapsto b \\ d \mapsto c. \end{cases}$$

As the exponent vectors $(2, 0, 0, 0)$, $(0, 1, 1, 1)$, $(0, 0, 2, 0)$, and $(0, 0, 0, 2)$ of the relations a^2 , bcd , c^2 , and d^2 , respectively, are \mathbb{Z} -linearly independent forming a basis for the subgroup U in (2), we can modify this presentation so that the relations become

$$a^2, c^2, d^2, bcd, b^2(bcd)^{-2}c^2d^2, \sigma^k((ad)^4a^{-4}d^{-4}), \sigma^k((adacac)^4a^{-12}c^{-8}d^{-4}), \quad (3)$$

for every $k \in \mathbb{N}_0$. Since the L -presentation is invariant, the images $\sigma^k(a^{-4}d^{-4})$ and $\sigma^k(a^{-12}c^{-8}d^{-4})$ are consequences. Hence, the invariant finite L -presentation

$$\langle \{a, b, c, d\} \mid \{b^2(bcd)^{-2}c^2d^2\} \cup \{a^2, c^2, d^2, bcd\} \mid \{\sigma\} \mid \mathcal{R}' \rangle,$$

where $\mathcal{R}' = \{(ad)^4a^{-4}d^{-4}, (adacac)^4a^{-12}c^{-8}d^{-4}\}$, defines the Grigorchuk group \mathfrak{G} and, as $\mathfrak{G}/\mathfrak{G}' \cong \mathbb{Z}_2^3$, it has the form as claimed in Theorem 1.

4 A generating set for the Schur multiplier

Let G be a finitely generated group. We will use the results of Theorem 1 and the Hopf formula to give a generating set for the Schur multiplier of G if G is invariantly finitely L -presented. Suppose that F is a finitely generated free group and R is a normal subgroup of F so that $G \cong F/R$. Then $F/[R, F]$ is a central extension of $R/[R, F]$ by the group G and the subgroup $R/[R, F]$

contains $(R \cap F')/[R, F]$. By the Hopf formula, the latter subgroup is isomorphic to the Schur multiplier of G . Further, the subgroup $R/[R, F]$ decomposes as follows.

Proposition 1 *Let $G \cong F/R$ with a finitely generated free group F . Then we have that*

$$R/[R, F] \cong \mathbb{Z}^{\text{rk}(F) - \mathfrak{h}(G/G')} \oplus M(G).$$

Proof. The factor RF'/F' is free abelian with torsion-free rank $\text{rk}(F) - \mathfrak{h}(G/G')$. Since $RF'/F' \cong R/(R \cap F')$ is free abelian, the subgroup $(R \cap F')/[R, F]$ has a free abelian complement of rank $\text{rk}(F) - \mathfrak{h}(G/G')$ and thus, the central subgroup $R/[R, F]$ decomposes as claimed above. \square

As $R/[R, F]$ is central in $F/[R, F]$, it is generated by the images of the normal generators of R . Thus, in particular, if R is finitely generated as normal subgroup (that is, if G is finitely presentable), then $R/[R, F]$ is a finitely generated abelian group and so is its subgroup $(R \cap F')/[R, F]$.

If G is finite, then $R/[R, F]$ is an abelian subgroup with finite index in $F/[R, F]$. A finite presentation for $F/[R, F]$ can be obtained from a finite presentation of G . Then the Reidemeister-Schreier algorithm yields a finite presentation for $R/[R, F]$ from which the isomorphism type of $M(G)$ is obtained easily.

If G is polycyclic, then it is finitely presentable and hence, the group $F/[R, F]$ is an extension of a finitely generated abelian group by a polycyclic group. In particular, $F/[R, F]$ is polycyclic in this case. A consistent polycyclic presentation for $F/[R, F]$ can be computed with the algorithm in [10]. This polycyclic presentation enables us to read off the isomorphism type of $R/[R, F]$ and, by Proposition 1, the isomorphism type of $M(G)$. If G is finitely generated and nilpotent of class c , then $F/[R, F]$ is nilpotent of class at most $c + 1$. If G is given by a weighted nilpotent presentation, then the algorithm in [23] computes a weighted nilpotent presentation for $F/[R, F]$.

We now consider the case of an invariantly finitely L -presented group G . Even though its Schur multiplier is not computable in general, the following theorem yields a generating set for $M(G)$ as subgroup of $R/[R, F]$.

Theorem 3 *Let $\langle \mathcal{X} \mid \mathcal{Q}' \cup \mathcal{B} \mid \Phi \mid \mathcal{R}' \rangle$ be an invariant finite L -presentation of G as provided by Theorem 1. Further let $\pi: F \rightarrow F/[R, F], x \mapsto \bar{x}$ denote the natural homomorphism. Then we have that*

$$M(G) \cong \langle \bar{q}, \bar{r}^\varphi \mid q \in \mathcal{Q}', r \in \mathcal{R}', \varphi \in \Phi^* \rangle.$$

Proof. Clearly, $R/[R, F]$ is generated by the images of $\mathcal{Q}' \cup \mathcal{B} \cup \bigcup_{\varphi \in \Phi^*} (\mathcal{R}')^\varphi$. As the relators in $\mathcal{Q}' \cup \mathcal{R}'$ are contained in F' , it holds that

$$\{\bar{q}, \bar{r}^\varphi \mid q \in \mathcal{Q}', r \in \mathcal{R}', \varphi \in \Phi^*\} \subseteq (R \cap F')/[R, F]. \quad (4)$$

We are left with the relators in \mathcal{B} . Recall that we have $|\mathcal{B}| = \text{rk}(F) - \mathfrak{h}(G/G')$. Hence, the images $\{\bar{r} \mid r \in \mathcal{B}\}$ generate a free abelian complement to the Schur multiplier $(R \cap F')/[R, F]$ in $R/[R, F]$. Therefore, the images in (4) necessarily generate $(R \cap F')/[R, F]$. \square

As the group G in Theorem 3 is invariantly L -presented, for every endomorphism $\varphi \in \Phi$, we have $R^\varphi \subseteq R$ and $[R, F]^\varphi \subseteq [R, F]$. Therefore, every $\varphi \in \Phi$ also induces an endomorphism of $F/[R, F]$ which fixes the subgroup $R/[R, F]$. Further, as F' is fully-invariant, every such φ induces an endomorphism $\bar{\varphi}$ of $(R \cap F')/[R, F]$. This yields that

$$M(G) \cong \langle \bar{q}, \bar{r}^{\bar{\varphi}} \mid q \in \mathcal{Q}', r \in \mathcal{R}', \varphi \in \Phi^* \rangle$$

and hence, the free monoid Φ^* induces a Φ^* -module structure on the Schur multiplier $M(G)$ in a natural way:

Lemma 4 *Let $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ be an invariant finite L -presentation. Then the Schur multiplier $M(G)$ is finitely generated as a Φ^* -module.*

In particular, the Schur multiplier $M(G)$ has the form $A \oplus \bigoplus_{\Phi^*} B$ with finitely generated abelian groups A and B ; see [1].

We proceed with Example 2 by describing a generating set for the Schur multiplier of the Grigorchuk group as provided by Theorem 3; cf. [15].

Example 5 *Consider the invariant finite L -presentation of the Grigorchuk group \mathfrak{G} as determined in Example 2. Then the images of*

$$b^2(bcd)^{-2}c^2d^2, \sigma^k((ad)^4a^{-4}d^{-4}), \sigma^k((adacac)^4a^{-12}c^{-8}d^{-4}), \quad \text{with } k \in \mathbb{N}_0, \quad (5)$$

in $F/[R, F]$, generate the subgroup $(R \cap F')/[R, F]$. The images in $F/[R, F]$ of the relations a^2 , c^2 , d^2 , and bcd generate a free abelian complement to the Schur multiplier $(R \cap F')/[R, F]$ in $R/[R, F]$.

5 Approximating the Schur multiplier

We finally describe our algorithm for approximating the Schur multiplier of an invariantly finitely L -presented group G . Let $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ be an invariant finite L -presentation defining the group F/R so that $G \cong F/R$. Then G is finitely generated and hence, its lower central series quotient $G/\gamma_c G$ is polycyclic. The nilpotent quotient algorithm in [2] computes a weighted nilpotent presentation for $G/\gamma_c G$ together with the natural homomorphism $\pi: F \rightarrow G/\gamma_c G$. In [23], Nickel described a covering-algorithm which, given a weighted nilpotent presentation for $G/\gamma_c G$ and the homomorphism π , computes a polycyclic presentation for $F/[R\gamma_c F, F]$ together with the natural homomorphism $\bar{\pi}: F \rightarrow F/[R\gamma_c F, F]$. The homomorphism $\bar{\pi}$ induces the homomorphism $\varphi_c: M(G) \rightarrow M(G/\gamma_c G)$ as follows: By Theorem 1, the group G has an invariant finite L -presentation of the form

$$\langle \mathcal{X} \mid \mathcal{Q}' \cup \mathcal{B} \mid \Phi \mid \mathcal{R}' \rangle, \quad \text{with } \mathcal{Q}', \mathcal{R}' \subset F'$$

and $|\mathcal{B}| = |\mathcal{X}| - \mathfrak{h}(G/G')$. Now, by Theorem 3, the images of $\mathcal{Q}' \cup \bigcup_{\varphi \in \Phi^*} (\mathcal{R}')^\varphi$ in $F/[R, F]$ generate the subgroup $(R \cap F')/[R, F]$. Similarly, the their images in $F/[R\gamma_c F, F]$ generate the subgroup $(R \cap F')[R\gamma_c F, F]/[R\gamma_c F, F]$. Since $[R\gamma_c F, F] = [R, F]\gamma_{c+1}F$, we have that

$$(R \cap F')[R\gamma_c F, F]/[R\gamma_c F, F] = (R\gamma_{c+1}F \cap F')/[R\gamma_c F, F].$$

The latter subgroup is contained in $(R\gamma_c F \cap F')/[R\gamma_c F, F]$ which is isomorphic to the Schur multiplier $M(G/\gamma_c G)$.

As the group G is invariantly L -presented, every $\varphi \in \Phi$ induces an endomorphism $\tilde{\varphi}$ of $R\gamma_c F/[R\gamma_c F, F]$. This yields, that the image of $M(G)$ in $M(G/\gamma_c G)$ has the form

$$\langle q^{\bar{p}}, (r^{\bar{\pi}})^{\tilde{\varphi}} \mid q \in \mathcal{Q}', r \in \mathcal{R}', \varphi \in \Phi^* \rangle. \quad (6)$$

This can be used to investigate the Φ^* -module structure of $M(G)$ by considering the finitely generated Dwyer quotients $M_c(G)$. In our algorithm, we use Hermite normal form computations in a spinning algorithm for computing a finite generating set of the subgroup in (6). We summarize our algorithm as follows: Write $G = F/R$.

DWYERQUOTIENT(G, c)

 Compute an invariant finite L -presentation as in Theorem 1.

 Compute a weighted nilpotent presentation for $G/\gamma_c G$
 together with the natural homomorphism $F \rightarrow G/\gamma_c G$.

 Compute a polycyclic presentation for the group $F/[R\gamma_c F, F]$
 together with the natural homomorphism $F \rightarrow F/[R\gamma_c F, F]$.

 Translate each $\varphi \in \Phi$ to an endomorphism of the group $F/[R\gamma_c F, F]$
 and restrict this endomorphism to $(R\gamma_{c+1} F \cap F')/[R\gamma_c F, F]$.

 Use the spinning algorithm to compute a finite generating set
 for the image $(R\gamma_{c+1} F \cap F')/[R\gamma_c F, F]$.

6 Applications

The algorithm described in the first part is available in the NQL-package [19] of the computer algebra system GAP; see [12]. We parallelized the algorithm in [2] to enlarge the possible depths in the lower central series reached in this section. We show the successful application of our algorithm to the following invariantly finitely L -presented *testbed groups* studied in [1] and [2]:

- The Grigorchuk group \mathfrak{G} , see [14], with its invariant finite L -presentation from [21]; see also [15] and Example 2;
- the twisted twin $\bar{\mathfrak{G}}$ of the Grigorchuk group, see [4], with its invariant finite L -presentation from [4];
- the Grigorchuk super-group $\tilde{\mathfrak{G}}$, see [3], with its invariant finite L -presentation from [1];
- the Basilica group Δ , see [18, 17], with its invariant finite L -presentation from [5]; and
- the Brunner-Sidki-Vieira group BSV, see [9], with its invariant finite L -presentation from [1].

In Section 6.3, we further applied our algorithm to several generalized Fabrykowski-Gupta groups: an infinite family of finitely L -presented groups Γ_p introduced in [16]. Invariant finite L -presentations for these groups were computed in [2].

6.1 Aspects of the implementation of our algorithm in GAP

Table 1 shows some performance data of the implementation of our algorithm in the NQL-package of the computer-algebra-system GAP. All timings displayed below have been obtained on an *Intel Pentium Core 2 Quad* with clock speed 2.83 GHz using a single core. We applied our algorithm with a time limit of two hours. Then the computations have been stopped and the total time used to compute a weighted nilpotent presentation for the quotient $G/\gamma_c G$ and the total time to compute the Dwyer quotient $M_c(G)$ have been listed. Every application completed within 1 GB of memory.

Table 1: Performance data of our implementation in GAP

G	c	Time (h:min) for		G	c	Time (h:min) for	
		$G/\gamma_{c+1}G$	$M_{c+1}(G)$			$G/\gamma_{c+1}G$	$M_{c+1}(G)$
\mathfrak{G}	90	1:47	0:07	Γ_4	71	1:50	0:07
\mathfrak{G}	54	1:44	0:09	Γ_5	55	1:40	0:04
$\tilde{\mathfrak{G}}$	44	1:32	0:13	Γ_7	46	1:40	0:03
Δ	42	1:31	0:16	Γ_8	56	1:54	0:06
BSV	35	1:10	0:21	Γ_9	61	1:44	0:06
Γ_3	75	1:46	0:04	Γ_{11}	35	1:54	0:02

We note that for the results shown in the remainder of this section we used a parallel version of the algorithm for computing $G/\gamma_{c+1}G$.

6.2 On the Dwyer quotients of the testbed-groups

The Dwyer quotient $M_c(G) = M(G)/\ker \varphi_c$ is a finitely generated abelian group and hence, it can be described by its abelian invariants or, if the group is p -elementary abelian, by its p -rank. Here the list (c_1, \dots, c_n) stands for the group $\mathbb{Z}_{c_1} \oplus \dots \oplus \mathbb{Z}_{c_n}$. For abbreviation, we will write $a^{[\ell]}$ if the term a occurs in ℓ consecutive places in a list. In the following we summarize our computational results for the testbed groups.

The Grigorchuk group \mathfrak{G} was shown in [14] to be an explicit counter-example to the general Burnside problem: it is a finitely generated infinite 2-torsion group. Furthermore, the Grigorchuk group is a first example of a group with an intermediate word-growth. In [21], Lysënok determined a first L -presentation for the group \mathfrak{G} ; see Example 2. Even though it was already proposed in [14] that the Grigorchuk group \mathfrak{G} is not finitely presentable, a proof was not derived until [15] where Grigorchuk explicitly computed the Schur multiplier of \mathfrak{G} : it is infinitely generated 2-elementary abelian. We have computed the Dwyer quotients $M_c(\mathfrak{G})$, for $1 \leq c \leq 301$. These quotients are 2-elementary abelian with the following 2-ranks

$$1, 2, 3^{[3]}, 5^{[6]}, 7^{[12]}, 9^{[24]}, 11^{[48]}, 13^{[96]}, 15^{[110]}.$$

This suggests the following conjecture.

Conjecture A *The Grigorchuk group \mathfrak{G} satisfies*

$$M_c(\mathfrak{G}) \cong \left\{ \begin{array}{ll} \mathbb{Z}_2 \text{ or } (\mathbb{Z}_2)^2, & \text{if } c = 1 \text{ or } c = 2, \text{ respectively} \\ (\mathbb{Z}_2)^{2m+3}, & \text{if } c \in \{3 \cdot 2^m, \dots, 3 \cdot 2^{m+1} - 1\} \end{array} \right\},$$

with $m \in \mathbb{N}_0$.

Further experiments suggest that the Schur multiplier of the Grigorchuk group \mathfrak{G} has the $\{\sigma\}^*$ -module structure, as given by Lemma 4, of the form $\mathbb{Z}_2 \oplus (\mathbb{Z}_2[\sigma])^2$ where σ fixes the first component.

The twisted twin $\bar{\mathfrak{G}}$ of the Grigorchuk group was introduced in [4]. It is invariantly finitely L -presented by

$$\langle \{a, b, c, d\} \mid \{a^2, b^2, c^2, d^2\} \mid \{\bar{\sigma}\} \mid \{[d^a, d], [d, c^a b], [d, (c^a b)^c], [d, (c^a b)^c], [c^a b, c b^a]\} \rangle$$

where $\bar{\sigma}$ is the free group endomorphism induced by the mapping

$$\bar{\sigma}: \begin{cases} a \mapsto c^a \\ b \mapsto d \\ c \mapsto b^a \\ d \mapsto c. \end{cases}$$

We have computed the Dwyer quotients $M_c(\bar{\mathfrak{G}})$, for $1 \leq c \leq 144$. These quotients are 2-elementary abelian with the following 2-ranks

$$2, 5, 7, 8^{[2]}, 11^{[2]}, 12^{[4]}, 15^{[4]}, 16^{[8]}, 19^{[8]}, 20^{[16]}, 23^{[16]}, 24^{[32]}, 27^{[32]}, 28^{[17]}.$$

This suggests the following conjecture.

Conjecture B *The twisted twin $\bar{\mathfrak{G}}$ of the Grigorchuk group satisfies*

$$M_c(\bar{\mathfrak{G}}) \cong \left\{ \begin{array}{ll} (\mathbb{Z}_2)^2, (\mathbb{Z}_2)^5, \text{ or } (\mathbb{Z}_2)^7, & \text{if } c = 1, c = 2, \text{ or } c = 3, \text{ resp.} \\ (\mathbb{Z}_2)^{4(m+1)+4}, & \text{if } c \in \{2^{m+2}, \dots, 2^{m+2} + 2^{m+1} - 1\} \\ (\mathbb{Z}_2)^{4(m+1)+7}, & \text{if } c \in \{2^{m+2} + 2^{m+1}, \dots, 2^{m+3} - 1\} \end{array} \right\},$$

with $m \in \mathbb{N}_0$.

Further experiments suggest that the Schur multiplier of $\bar{\mathfrak{G}}$ has the $\{\bar{\sigma}\}^*$ -module structure, as given by Lemma 4, of the form $(\mathbb{Z}_2[\bar{\sigma}])^4$; for a proof see [4].

The Grigorchuk super-group $\tilde{\mathfrak{G}}$ was introduced in [3]. It contains the Grigorchuk group \mathfrak{G} as an infinite-index subgroup and it is another example of a group with an intermediate word-growth. In [1], it was shown that $\tilde{\mathfrak{G}}$ admits the invariant finite L -presentation $\langle \{\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\} \mid \emptyset \mid \{\tilde{\sigma}\} \mid \mathcal{R} \rangle$ where

$$\mathcal{R} = \{a^2, [\tilde{b}, \tilde{c}], [\tilde{c}, \tilde{c}^a], [\tilde{c}, \tilde{d}^a], [\tilde{d}, \tilde{d}^a], [\tilde{c}^{\tilde{a}\tilde{b}}, (\tilde{c}^{\tilde{a}\tilde{b}})^a], [\tilde{c}^{\tilde{a}\tilde{b}}, (\tilde{d}^{\tilde{a}\tilde{b}})^a], [\tilde{d}^{\tilde{a}\tilde{b}}, (\tilde{d}^{\tilde{a}\tilde{b}})^a]\}$$

and $\tilde{\sigma}$ is the free group endomorphism induced by the mapping

$$\tilde{\sigma}: \begin{cases} a \mapsto \tilde{a}\tilde{b}a \\ \tilde{b} \mapsto \tilde{d} \\ \tilde{c} \mapsto \tilde{b} \\ \tilde{d} \mapsto \tilde{c}. \end{cases}$$

The Schur multiplier of the group $\tilde{\mathfrak{G}}$ is still unknown. We have computed the Dwyer quotients $M_c(\tilde{\mathfrak{G}})$, for $1 \leq c \leq 232$. These quotients are 2-elementary abelian with the following 2-ranks

$$3, 6, 7, 9^{[2]}, 11^{[2]}, 13^{[4]}, 15^{[4]}, 17^{[8]}, 19^{[8]}, 21^{[16]}, 23^{[16]}, 25^{[32]}, 27^{[32]}, 29^{[64]}, 31^{[41]}.$$

This suggests the following conjecture.

Conjecture C *The Grigorchuk super-group $\tilde{\mathfrak{G}}$ satisfies*

$$M_c(\tilde{\mathfrak{G}}) \cong \left\{ \begin{array}{ll} (\mathbb{Z}_2)^3, (\mathbb{Z}_2)^6, \text{ or } (\mathbb{Z}_2)^7, & \text{if } c = 1, 2, \text{ or } 3, \text{ respectively} \\ (\mathbb{Z}_2)^{4m+5}, & \text{if } c \in \{2 \cdot 2^m, \dots, 3 \cdot 2^m - 1\} \\ (\mathbb{Z}_2)^{4m+7}, & \text{if } c \in \{3 \cdot 2^m, \dots, 2 \cdot 2^{m+1} - 1\} \end{array} \right\},$$

with $m \in \mathbb{N}$.

Further experiments suggest that the Schur multiplier of the Grigorchuk super-group has the $\{\tilde{\sigma}\}^*$ -module structure, as given by Lemma 4, of the form $(\mathbb{Z}_2)^3 \oplus (\mathbb{Z}_2[\tilde{\sigma}])^4$, where $\tilde{\sigma}$ cyclically permutes the first component.

The Basilica group Δ was introduced in [17, 18] as a torsion-free group defined by a three-state automaton. Bartholdi and Virág [5] computed the following invariant finite L -presentation:

$$\Delta \cong \langle \{a, b\} \mid \emptyset \mid \{\sigma\} \mid \{[a, a^b]\} \rangle$$

where σ is the free group endomorphism induced by the mapping

$$\sigma: \begin{cases} a \mapsto b^2 \\ b \mapsto a. \end{cases}$$

We have computed the Dwyer quotients $M_c(\Delta)$, for $1 \leq c \leq 103$. These quotients satisfy the following conjecture.

Conjecture D *The Basilica group Δ satisfies*

$$M_c(\Delta) \cong \mathbb{Z}^2 \oplus \bigoplus_{\ell \in \mathbb{N}} \mathcal{A}_\ell(c), \quad \text{for each } c \geq 2,$$

where the groups $\mathcal{A}_\ell(c)$ are given as follows:

$$\mathcal{A}_1(c) = \left\{ \begin{array}{ll} 0, & \text{if } c \in \{1, \dots, 5\} \\ \mathbb{Z}_{2^{2(m+1)}}, & \text{if } c \in \{2m+6, 2m+7\} \end{array} \right\}$$

and

$$\mathcal{A}_\ell(c) = \left\{ \begin{array}{ll} 0, & \text{if } c \in \{1, \dots, 3 \cdot 2^{\ell+1} - 1\} \\ \mathbb{Z}_{2^{2m+1}}, & \text{if } c \in \{(3+m) \cdot 2^{\ell+1}, \dots, (3+m) \cdot 2^{\ell+1} + 2^{\ell-1} - 1\} \\ \mathbb{Z}_{2^{2m+2}}, & \text{if } c \in \{(3+m) \cdot 2^{\ell+1} + 2^{\ell-1}, \dots, (4+m) \cdot 2^{\ell+1} - 1\} \end{array} \right\}$$

with $m \in \mathbb{N}_0$. Hence, the Basilica group Δ is not finitely presentable.

The Brunner-Sidki-Vieira group BSV was introduced in [9] as a just-non-solvable, torsion-free group acting on the binary tree. The authors also gave the following invariant finite L -presentation:

$$\text{BSV} \cong \langle \{a, b\} \mid \emptyset \mid \{\varepsilon\} \mid \{[b, b^a], [b, b^{a^3}]\} \rangle$$

where ε is the free group endomorphism induced by the mapping

$$\varepsilon: \begin{cases} a & \mapsto a^2 \\ b & \mapsto a^2 b^{-1} a^2. \end{cases}$$

We have computed the Dwyer quotients $M_c(\text{BSV})$, for $1 \leq c \leq 53$. These quotients satisfy the following conjecture.

Conjecture E *The Brunner-Sidki-Vieira group BSV satisfies*

$$M_c(\text{BSV}) \cong \mathbb{Z}^2 \oplus \mathcal{A}(c) \oplus \bigoplus_{\ell \in \mathbb{N}} \mathcal{B}_\ell(c) \oplus \bigoplus_{\ell \in \mathbb{N}} \mathcal{C}_\ell(c), \quad \text{for each } c \geq 2,$$

where the groups $\mathcal{A}(c)$, $\mathcal{B}_\ell(c)$, and $\mathcal{C}_\ell(c)$ are given as follows:

$$\mathcal{A}(c) = \begin{cases} 0, & \text{if } c \in \{1, \dots, 3\} \\ \mathbb{Z}_{2^{2m+1}}, & \text{if } c \in \{2m+4, 2m+5\} \end{cases}$$

with $m \in \mathbb{N}_0$. Additionally, for each $\ell \in \mathbb{N}$, we have

$$\mathcal{B}_\ell(c) = \begin{cases} 0, & \text{if } c \in \{1, \dots, 5 \cdot 2^{\ell-1} - 1\} \\ \mathbb{Z}_{2^{4m+1}}, & \text{if } c \in \{2^{\ell+2}m + 5 \cdot 2^{\ell-1}, \dots, 2^{\ell+2}m + 6 \cdot 2^{\ell-1} - 1\} \\ \mathbb{Z}_{2^{4m+2}}, & \text{if } c \in \{2^{\ell+2}m + 6 \cdot 2^{\ell-1}, \dots, 2^{\ell+2}m + 10 \cdot 2^{\ell-1} - 1\} \\ \mathbb{Z}_{2^{4m+4}}, & \text{if } c \in \{2^{\ell+2}m + 10 \cdot 2^{\ell-1}, \dots, 2^{\ell+2}m + 13 \cdot 2^{\ell-1} - 1\} \end{cases}$$

and

$$\mathcal{C}_\ell(c) = \begin{cases} 0, & \text{if } c \in \{1, \dots, 9 \cdot 2^{\ell-1} - 1\} \\ \mathbb{Z}_{2^{4m+1}}, & \text{if } c \in \{2^{\ell+2}m + 9 \cdot 2^{\ell-1}, \dots, 2^{\ell+2}m + 12 \cdot 2^{\ell-1} - 1\} \\ \mathbb{Z}_{2^{4m+2}}, & \text{if } c \in \{2^{\ell+2}m + 12 \cdot 2^{\ell-1}, \dots, 2^{\ell+2}m + 14 \cdot 2^{\ell-1} - 1\} \\ \mathbb{Z}_{2^{4m+3}}, & \text{if } c \in \{2^{\ell+2}m + 14 \cdot 2^{\ell-1}, \dots, 2^{\ell+2}m + 16 \cdot 2^{\ell-1} - 1\} \\ \mathbb{Z}_{2^{4m+4}}, & \text{if } c \in \{2^{\ell+2}m + 16 \cdot 2^{\ell-1}, \dots, 2^{\ell+2}m + 17 \cdot 2^{\ell-1} - 1\} \end{cases}$$

with $m \in \mathbb{N}_0$. Hence, the Brunner-Sidki-Vieira group BSV is not finitely presentable.

6.3 On the Dwyer quotients of some Fabrykowski-Gupta groups

The Fabrykowski-Gupta group Γ_3 was introduced in [11] as an example of a group with an intermediate word-growth. For every positive integer d , Grigorchuk [16] described a generalization Γ_d of the Fabrykowski-Gupta group Γ_3 . A rather longish invariant finite L -presentation was computed in [2]. Further, it was shown that, if d is not a prime-power, the group Γ_d has a maximal nilpotent quotient. This latter quotient is isomorphic to the maximal nilpotent quotient of the wreath product $\mathbb{Z}_d \wr \mathbb{Z}_d$. We therefore consider only those groups Γ_d which admit a ‘rich’ lower central series; that is, the index d is a prime-power.

Let $d \in \{3, 5, 7, 11\}$ be a prime. Then the Dwyer quotients $M_c(\Gamma_d)$ are d -elementary abelian with the following d -ranks.

d	$\text{rk}(M_c(\Gamma_d))$
3	$0^{[2]}, 1^{[3]}, 2^{[0]}, 3^{[9]}, 4^{[1]}, 5^{[26]}, 6^{[4]}, 7^{[77]}, 8^{[13]}, 9^{[12]}$
5	$0^{[1]}, 1^{[4]}, 2^{[2]}, 3^{[20]}, 4^{[10]}, 5^{[100]}, 6^{[2]}$
7	$0^{[1]}, 1^{[2]}, 2^{[6]}, 3^{[2]}, 4^{[14]}, 5^{[42]}, 6^{[14]}, 7^{[34]}$
11	$0^{[1]}, 1^{[2]}, 2^{[2]}, 3^{[2]}, 4^{[10]}, 5^{[2]}, 6^{[22]}, 7^{[22]}, 8^{[22]}, 9^{[27]}$

As noted by Laurent Bartholdi and Olivier Siegenthaler, there is a pattern in the ranks of the Dwyer quotients $M_c(\Gamma_d)$. For example, it may holds that

$$M_c(\Gamma_5) \cong \left\{ \begin{array}{ll} 0, & \text{if } c = 0 \\ \mathbb{Z}_5^{2m+1}, & \text{if } c \in \{2 + \frac{3}{2}(5^m - 1), \dots, 1 + \frac{3}{2}(5^m - 1) + 4 \cdot 5^m\} \\ \mathbb{Z}_5^{2m+2}, & \text{if } c \in \{2 + \frac{3}{2}(5^m - 1) + 4 \cdot 5^m, \dots, 1 + \frac{3}{2}(5^{m+1} - 1)\} \end{array} \right\}$$

for $m \in \mathbb{N}_0$. This suggests the following conjecture.

Conjecture F *Let d be a prime. Then the Schur multiplier of Γ_d , modulo the Dwyer-kernel, is infinitely generated d -elementary abelian.*

Finally, we summarize our results for $M_c(\Gamma_d)$ for $d \in \{4, 8, 9\}$. The abelian invariants of the Dwyer quotients $M_c(\Gamma_d)$ are as follows.

d	$M_c(\Gamma_d)$
4	$(1)^{[1]} (2)^{[1]} (2, 2)^{[1]} (2, 4)^{[4]} (2, 2, 2, 4)^{[1]}$
	$(2, 2, 2, 2, 4)^{[4]} (2, 2, 2, 2, 4, 4)^{[16]} (2, 2, 2, 2, 2, 4, 4)^{[1]} (2, 2, 2, 2, 2, 2, 4, 4)^{[3]}$
	$(2, 2, 2, 2, 2, 2, 4, 4)^{[16]} (2, 2, 2, 2, 2, 2, 4, 4, 4)^{[64]} (2, 2, 2, 2, 2, 2, 2, 4, 4, 4)^{[5]}$
	$(2, 2, 2, 2, 2, 2, 2, 4, 4, 4)^{[11]} (2, 2, 2, 2, 2, 2, 2, 2, 4, 4, 4)^{[26]}$
8	$(1)^{[1]} (8)^{[2]} (4, 8)^{[3]} (2, 4, 8)^{[4]} (2, 8, 8)^{[1]} (2, 2, 8, 8)^{[2]}$
	$(2, 2, 2, 8, 8)^{[2]} (2, 2, 4, 8, 8)^{[2]} (2, 4, 4, 8, 8)^{[2]} (2, 4, 8, 8, 8)^{[2]}$
	$(2, 8, 8, 8, 8)^{[8]} (2, 2, 8, 8, 8, 8)^{[4]} (2, 4, 8, 8, 8, 8)^{[20]} (2, 2, 4, 8, 8, 8, 8)^{[32]}$
	$(2, 2, 8, 8, 8, 8, 8)^{[7]} (2, 2, 2, 8, 8, 8, 8, 8)^{[16]} (2, 2, 2, 2, 8, 8, 8, 8, 8)^{[16]}$
9	$(2, 2, 2, 4, 8, 8, 8, 8, 8)^{[16]} (2, 2, 4, 4, 8, 8, 8, 8, 8)^{[3]}$
	$(1)^{[1]} (9)^{[2]} (3, 9)^{[2]} (3, 3, 9)^{[4]} (3, 9, 9)^{[2]}$
	$(9, 9, 9)^{[2]} (3, 9, 9, 9)^{[2]} (3, 3, 9, 9, 9)^{[4]} (3, 9, 9, 9, 9)^{[2]}$
	$(9, 9, 9, 9, 9)^{[12]} (3, 9, 9, 9, 9, 9)^{[18]} (3, 3, 9, 9, 9, 9, 9)^{[36]}$
	$(3, 9, 9, 9, 9, 9, 9)^{[18]} (9, 9, 9, 9, 9, 9, 9, 9)^{[17]} (3, 9, 9, 9, 9, 9, 9, 9)^{[12]}$

Again, these computational results suggest that the groups Γ_d are not finitely presentable. Further, the exponent of $M_c(\Gamma_d)$ is most likely the index d itself.

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